Colour-Permuting and Colour-Preserving Automorphisms

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based on joint work with Ademir Hujdurovič, Klavdija Kutnar, and Dave Witte Morris

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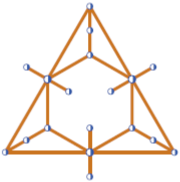

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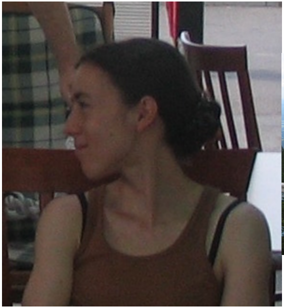
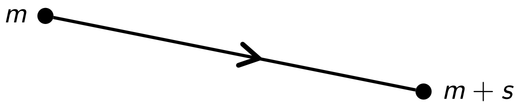


Definition of a Circulant Graph

Definition

$Circ(n; S)$ is the digraph whose vertices are the elements of $\mathbb{Z}_n$, with m adjacent to $m + s$ iff $s \in S$.



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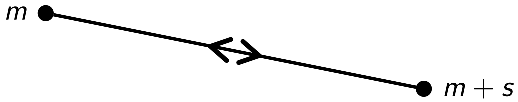
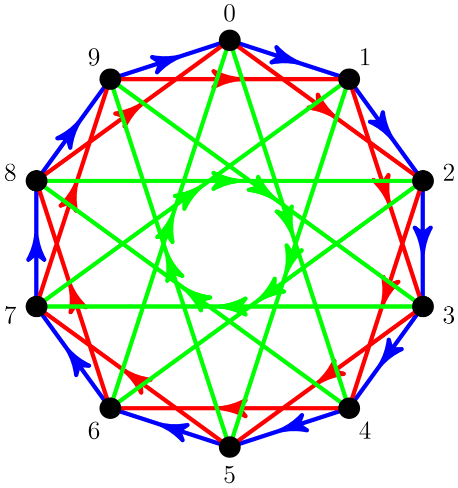


Example



$n = 10$, $S = \{1, 2, 4\}$

Definition of a Cayley Graph

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Automorphisms from the group action

Notice that for any $h \in G$, left-multiplying every vertex by h is a graph automorphism. This action is called the left translation by h. The group of all such actions is called the left-regular representation of G.

Observe

that if α is a group automorphism of G, then α determines an isomorphism from $\mathrm{Cay}(G; S)$ to $\mathrm{Cay}(G; \alpha(S))$. This isomorphism is a graph automorphism of $\mathrm{Cay}(G; S)$, precisely if $\alpha(S) = S$.

Thus, every Cayley graph has a natural, affine group of automorphisms that comes from the group action: $G_L \rtimes \mathrm{Aut}(G; S)$.

Normal Cayley graphs

Definition (M.Y. Xu)

A Cayley (di)graph $\Gamma = \mathrm{Cay}(G; S)$ *is normal if the left-regular representation of* G *is normal in Aut(*Γ*).*

Theorem (Godsil, 1981)

A Cayley (di)graph is normal if and only if its automorphism group is $G_L \rtimes \mathrm{Aut}(G; S)$.

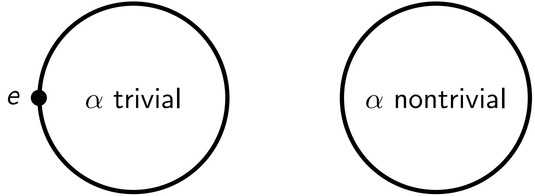
Thus, normal Cayley graphs are Cayley graphs in which every automorphism is one of the natural automorphisms that arises from the group action.

When are these affine transformations all we get?

For every group G with $|G| > 3$, there is some Cayley graph that has automorphisms that don't come from the group action: the empty graph, since S_n is not affine for $n > 3$.

Obvious obstruction: connectedness.

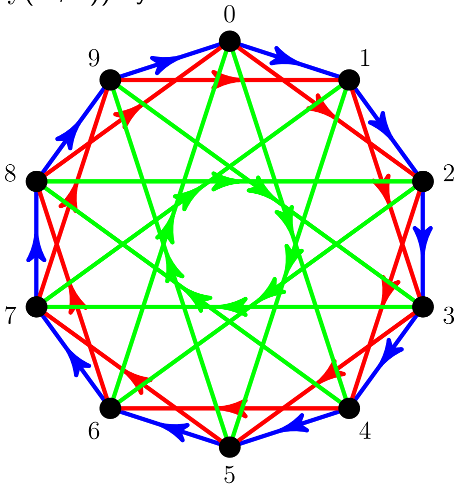
Not (in general) a group automorphism:



The complete graph is also a problem. Is there a natural condition we can make on graph automorphisms, that ensures that all Cayley graph automorphisms satisfying this restriction, are affine?

Notice...

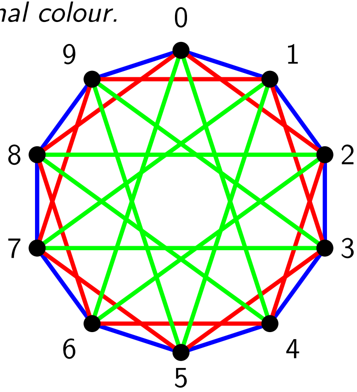
there is a natural colouring of the edges of $\mathrm{Circ}(n; S)$ (or more generally of $\mathrm{Cay}(G; S)$) by the elements of S.



$n = 10$, $S = \{1, 2, 4\}$

Definition

We say that an automorphism of a graph is colour-preserving if it fixes each of the colours given in a particular colouring.
We say that an automorphism of a graph is colour-permuting if whenever two edges have the same colour, their images also have the same colour. Note that their images may not be the same colour as the original colour.



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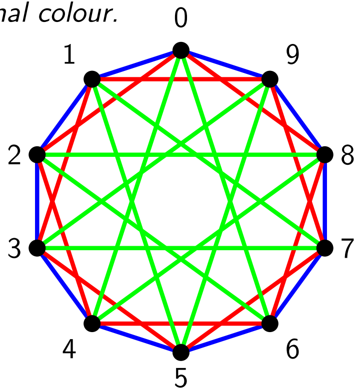
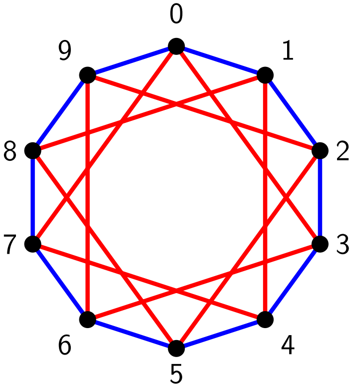
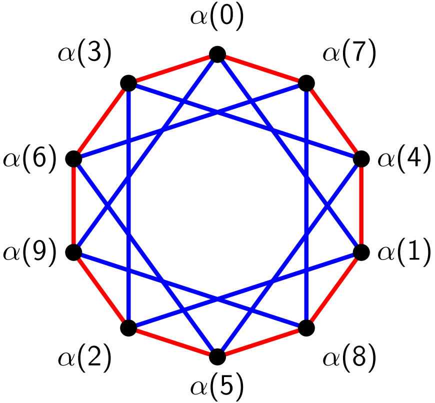


Observe

that if α is a group automorphism of G, then $\alpha(gs) = \alpha(g)\alpha(s)$, so the colour s maps to the colour $\alpha(s)$. Thus, such an automorphism is at least colour-permuting. Eg.

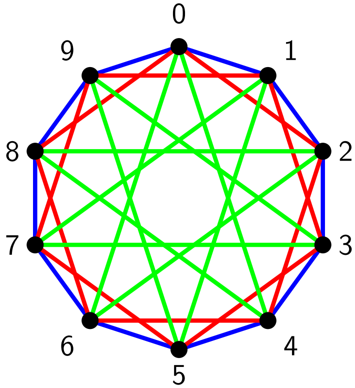


$\mathrm{Cay}(\mathbb{Z}_{10}; \{1, 9, 3, 7\})$



$\mathrm{Cay}(\mathbb{Z}_{10}; \{\alpha(1), \alpha(9), \alpha(3), \alpha(7)\})$

Also observe that left translation by any element of G is colour-preserving. So the graph automorphisms that come from the left-regular representation of G, are all colour-preserving. Eg.



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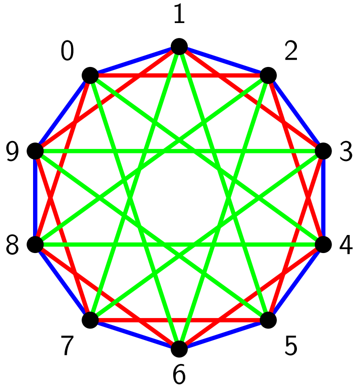

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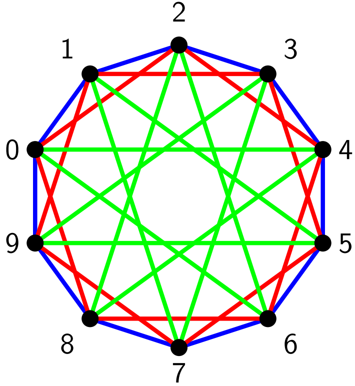

Proposition

In a connected Cayley digraph $Cay(G; S)$, any colour-permuting automorphism α that fixes the vertex e, is an automorphism of G.

Proof.

I will show that for any $g \in G$ and $s \in S$, $\alpha(gs) = \alpha(g)\alpha(s)$.
Suppose that the arc from g to gs is coloured red, so the arc from e to s is also red.

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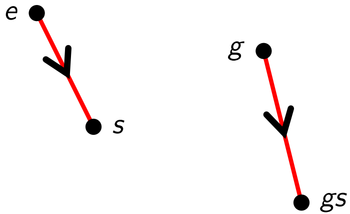

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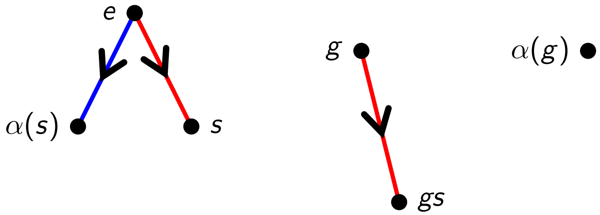

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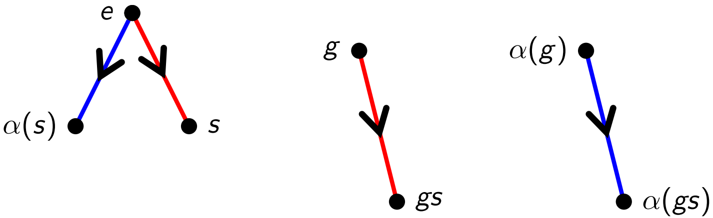

Notice...

in a graph (rather than a digraph), this proof won't work immediately, because $\alpha(s)$ could be s' or s'^{-1}.
So $\alpha(st)$ could be any one of

- $s't'$;
- $s't'^{-1}$;
- $s'^{-1}t$; or
- $s'^{-1}t'^{-1}$.

However, the proof will work if every element of S is an involution.

Also...

The condition of connectedness is necessary.

Question [Conder, Pisanski, Žitnik]

For circulant graphs, is a colour-permuting automorphism of the graph that fixes the identity vertex, necessarily an automorphism of the group? (i.e. a multiplier)



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This question arose in the context of studying the structure and automorphism groups of GI-graphs, which are a generalisation of both generalised Petersen graphs; and the Foster census I-graphs, but seemed of interest in its own right.

Answer [M., 2012^{+}]

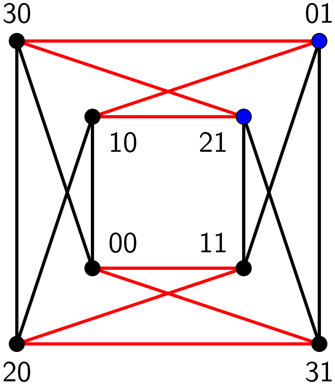
Yes (for connected circulants).

$\mathbb{Z}_4 \times \mathbb{Z}_2$

Let's start by studying the smaller class of colour-preserving automorphisms.

Theorem (Verret, 2014)

There is a Cayley graph on $\mathbb{Z}_4 \times \mathbb{Z}_2$ with a colour-preserving graph automorphism that is not a group automorphism: $Cay(\mathbb{Z}_4 \times \mathbb{Z}_2; \{10, 30, 11, 31\})$.



Definition of CCA

Definition

A Cayley graph is CCA (has the Cayley Colour Automorphism *property) if all of its colour-preserving automorphisms are "affine," i.e. composed from a group automorphism of G and left translation by an element of G (an element of the left-regular representation of G).*

Definition

A group G is CCA if every connected *Cayley graph on G is CCA.*

So we have seen:

- cyclic groups are CCA;
- $\mathbb{Z}_4 \times \mathbb{Z}_2$ is not CCA.

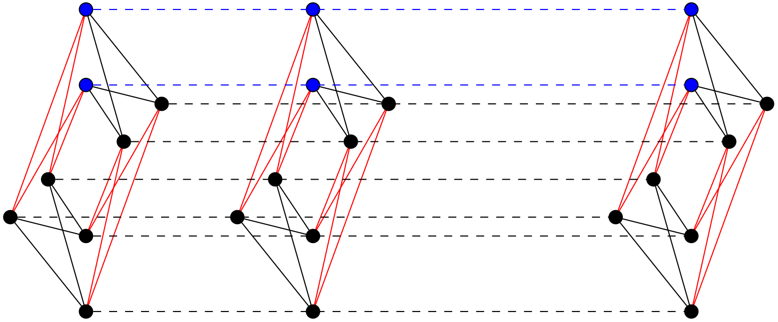
CCA and direct products

Theorem

If G is not CCA, and H is any group, then $G \times H$ is not CCA.

Proof.

Let $\Gamma = \mathrm{Cay}(G; S)$ be a non-CCA Cayley graph on G. Let $\Gamma' = \mathrm{Cay}(H; T)$ be any Cayley graph on H. Then $\Gamma \,\Box\, \Gamma'$ is a Cayley graph on $G \times H$. And $\Gamma \,\Box\, \Gamma'$ is not CCA. $\square$



CCA and direct products

Partial converse

If G and H are CCA and $\gcd(|G|, |H|) = 1$, then $G \times H$ is CCA.

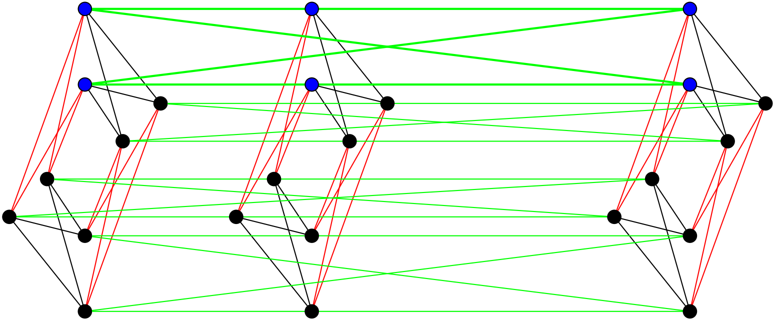
Our examples of $\mathbb{Z}_4 \times \mathbb{Z}_2$ and cyclic groups show us that the condition $\gcd(|G|, |H|) = 1$ is necessary.

Having a subgroup that is not CCA is not sufficient to ensure that a group is not CCA. In fact, $\mathbb{Z}_8 \times \mathbb{Z}_2$ is CCA.

$\mathbb{Z}_{2^n} \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $n \geq 2$

Theorem

$\mathrm{Cay}(\mathbb{Z}_{2^n} \times \mathbb{Z}_2 \times \mathbb{Z}_2; \{\pm(1,0,0), \pm(2^{n-2},1,0), \pm(2^{n-2},0,1)\})$ *is not CCA when $n \geq 2$.*



Abelian groups

Theorem

An abelian group is CCA if and only if it does not contain $\mathbb{Z}_4 \times \mathbb{Z}_2$ or $\mathbb{Z}_{2^n} \times \mathbb{Z}_2 \times \mathbb{Z}_2$ as a direct factor.
In particular, any abelian group whose order is not divisible by 8 is CCA.

Nonabelian groups

Definition

A generalised dihedral group D over an abelian group A is the group $\langle A, \tau \rangle$, where $\tau^2 = e$ and $a\tau = \tau a^{-1}$ for every $a \in A$.

Theorem

A generalised dihedral group over the abelian group A is CCA if and only if A is CCA.

Proof is similar to digraph proof. Notice this means that dihedral groups are always CCA.

Nonabelian groups - non-CCA

Definition

Let A be abelian of even order and choose an involution y of A. A *generalised dicyclic group* *over A is the group $\langle x, A\rangle$, where $x^2 = y$ and $ax = xa^{-1}$ for every $a \in A$.*
A *semidihedral* *group of order $16n$ is $\langle x, a\rangle$, where $a^{8n} = x^2 = e$ and $xa = a^{4n-1}x$.*

Theorem

The following groups are not CCA:

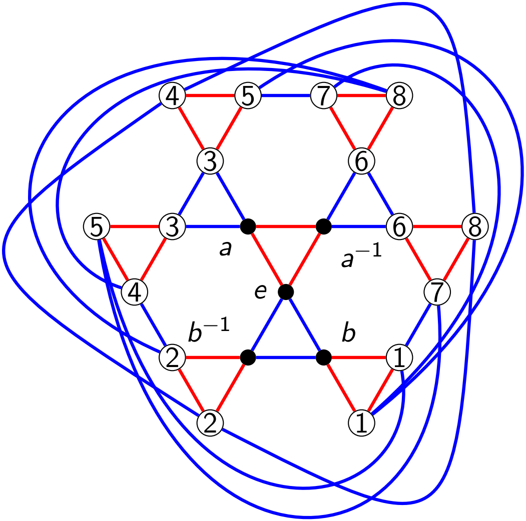
- *generalised dicyclic groups;*
- *semidihedral groups.*

The graphs look a lot like our previous examples.
In particular, there is a non-CCA group of order $4n$ for every $n \geq 3$.

A very different example

Theorem

Let $G = \langle a, b | a^3 = b^3 = e, (ab^{-1})^2 = b^{-1}a \rangle$*, the nonabelian group of order 21. Then* $\mathrm{Cay}(G; \{a^{\pm 1}, b^{\pm 1}\})$ *is not CCA.*



Some broad results

Theorem

The wreath product $\mathbb{Z}_m \wr \mathbb{Z}_n$ is not CCA when $m \geq 3$ and $n \geq 2$.

Theorem

A non-CCA group of odd order has a section isomorphic to at least one of:

- *the nonabelian group of order 21,*
- *or a semi-wreathed product $A \wr_\alpha \mathbb{Z}_n$, where A is a nontrivial elementary abelian group (of odd order) and $n > 1$.*

Theorem

There exists a non-CCA group of order n if and only if:

- *$n \geq 8$; and*
- *n is divisible by at least one of 4, 21, or $p^q q$.*

Strongly CCA

Question

What about colour-permuting automorphisms? There are more of them, so maybe some of them are not affine, even if all of the colour-preserving ones are?

Definition

A Cayley graph is strongly CCA if all of its colour-permuting automorphisms are affine. A group G is strongly CCA if every connected Cayley graph on G is CCA.

So we have seen:

- cyclic groups are strongly CCA; and
- any group that is not CCA, is not strongly CCA.

Are strongly CCA groups different from CCA groups?

Theorem

An abelian group is strongly CCA if and only if it is CCA.

Theorem

A group of odd order is strongly CCA if and only if it is CCA.

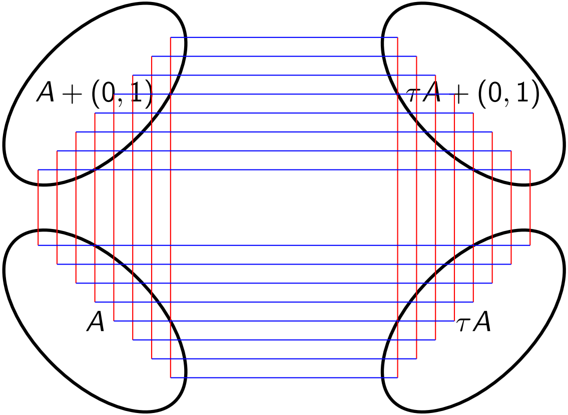
Proof.

It is clear that a strongly CCA group is CCA. Conversely, suppose G is CCA, and let Γ be any Cayley graph on G. Then G is normal in the group of colour-preserving automorphisms of Γ, since these are the affine transformations. In fact, G is characteristic in this group because it is the largest odd-order subgroup (details omitted). Also, the colour-preserving automorphism group is normal in the colour-permuting automorphism group, so G is normal in the colour-permuting automorphism group. Thus Γ is strongly CCA, so G is strongly CCA. $\square$

Generalised dihedral groups

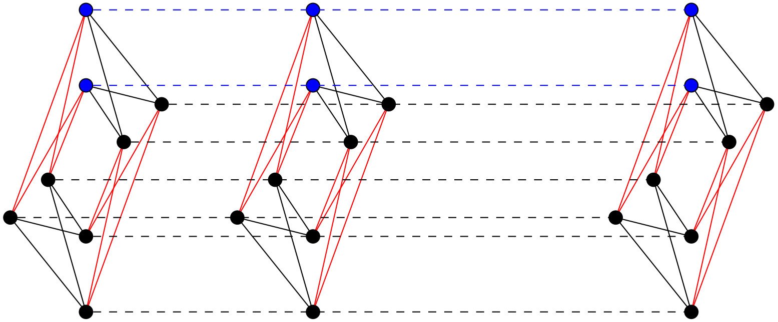
Theorem

Let A be any abelian group. The generalised dihedral group D over $A \times \mathbb{Z}_2$ is not *strongly CCA. In particular, let S be any generating set for A. Then $\mathrm{Cay}(D; S \cup \{(0,1), \tau\}$ is not strongly CCA.*

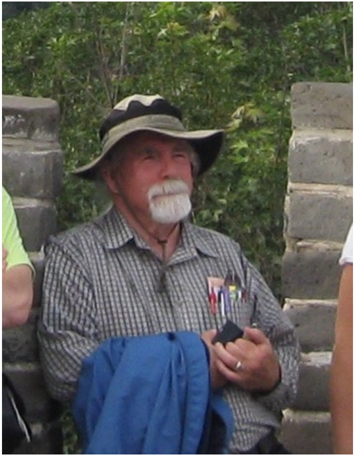


Recall

The same construction as before shows that if G is not strongly CCA and H is any group, then $G \times H$ is not strongly CCA.



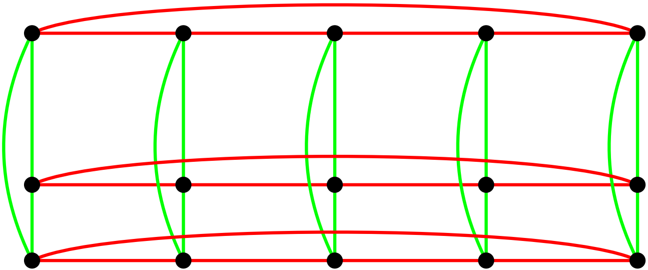
Linking Rings Structures



Linking Rings Structures

Definition (Potočnik, Wilson)

An LR (Linking Rings) structure is a 4-valent vertex transitive graph, whose edges have been coloured with 2 colours, red and green (say) such that every vertex is in a red cycle and a green cycle; the green cycles all have some fixed length k_1 and the red cycles all have some fixed length k_2; there are "swappers" at each vertex (automorphisms that fix the red/green cycle containing that vertex pointwise, while flipping the green/red cycle containing that vertex); and the group of colour-preserving automorphisms is vertex-transitive.



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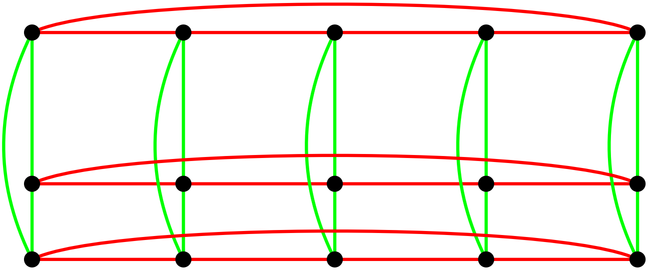
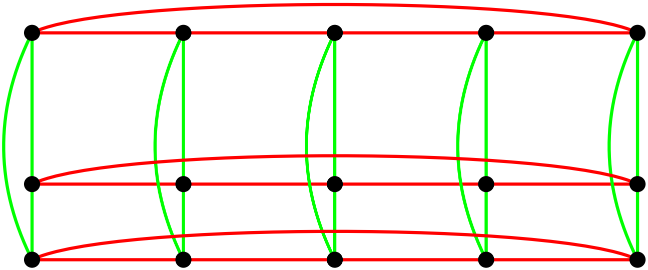


Why LR structures?

They are closely related to semisymmetric graphs of valency 4.
(Via the partial line graphs construction)

Linking Rings Structures



Relationship to our problem

An important method of constructing LR structures is via Cayley graphs. If the generators in the connection set are not involutions, then use the natural colouring. If some of the 4 generators are involutions, then some of the “natural” colour classes get merged. It is not easy to determine the existence of “swappers” in general; “Cayley swappers” are swappers that are colour-preserving group automorphisms.

Some Questions that Remain

Question

What other groups or graphs are CCA? Strongly CCA? Are graphs of “small” valency CCA, even on non-CCA groups?

Question

Are there other natural colourings for which we could ask this question? E.g. edge-orbits of a vertex-transitive action on a non-Cayley graph?

Thank you!

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